GENERALIZATION OF THE BEAM APPROACH TO PROBLEMS OF CRACK THEORY

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This note supplements a previous article [1] devoted to the problem of the motion of a crack along a bar, where it was assumed that the behavior of the bar is quite accurately described by the Bernoulli-Euler beam theory. Below (section 1) the formulation of the problem is extended to the two-dimensional case, i.e., to the case of crack propagation along the middle surface of a thin plate. This kind of problem can be reproduced experimentally using layered or laminated materials. In section 2 the beam formulation is generalized in another direction: in describing the behavior of a bar the effect of the shearing force on deflection and, moreover, the inertia of rotation of cross sections of the bar are taken into account. Consideration of these factors ensures the existence of a limiting crack propagation velocity. The equations presented were obtained by a variational method from the principle of least action; the calculations have been omitted because of their similarity to those in [1] and because, though clumsy, they are relatively elementary.

1. We consider the motion of a crack along the middle surface of a thin plate of thickness 2H, whose material possesses density ρ and elastic constants E (Young's modulus) and ν (Poisson's ratio), while the resistance to crack propagation is characterized by the surface energy density T. Let the crack lie in the xy-plane and at time t occupy a region D(t) bounded by the piecewise-smooth closed contour C(t). We denote the normal displacement of the neutral surface of one of the halves of the plate by u(x, t); we assume that the plate is loaded by an external force of density p(x, y, t) piecewise-continuous in D + C. Conditions of rigid restraint are assumed on the contour C(t) (n is the normal to the contour)

$$u(C)=0, \quad \frac{\partial u}{\partial n}(C)=0 \tag{1.1}$$

The expressions for the kinetic and potential energies take the form

$$K=\iint\limits_{D} \frac{\rho H}{2}\left(\frac{\partial u}{\partial t}\right)^{2} dx\, dy$$

$$\Pi=\iint\limits_{D} \frac{EH^{3}}{24(1-\nu^{2})}\left\{\Delta^{2} u+2(1-\nu)\left[\left(\frac{\partial^{2} u}{\partial x\, \partial y}\right)^{2}-\frac{\partial^{2} u}{\partial x^{2}} \frac{\partial^{2} u}{\partial y^{2}}\right]\right\} dx\, dy+\iint\limits_{D} T\, dx\, dy-\iint\limits_{D} pu\, dx\, dy$$

where Δ is a two-dimensional Laplacian.

The first integral in Π is equal to the flexural energy [2], the second to the work done on creating a new surface, while the third is the potential of the external forces. From the condition of stationarity of the action integral with conditions (1.1) at the moving boundary C(t) of the three-dimensional region $t \in [t_1, t_2]$, $(x, y) \in D(t)$ we obtain the following problem: we are required to find a function, continuous with partial derivatives up to second order with respect to t and up to fourth order with respect to x and y, and a contour C(t) such that with

$$\Delta^{2} u+\frac{1}{a_{1}^{2}} \frac{\partial^{2} u}{\partial t^{2}}=\frac{12(1-\nu^{2})}{EH^{3}} p, \quad a_{1}^{2}=\frac{EH^{3}}{12 \rho(1-\nu^{2})} \tag{1.2}$$

$$\frac{\partial^{2} u}{\partial n^{2}}(C)=A_{1}, \quad A_{1}=\left(\frac{24(1-\nu^{2}) T}{EH^{3}}\right)^{1/2} \tag{1.3}$$

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conditions (1.1) are satisfied and at t = 0 certain initial data compatible with (1.1) and (1.3). If we assume loading by concentrated forces and moments, it is sufficient to require only the continuity of the displacement with first partial derivatives. Conditions (1.1) and (1.3) ensure the smoothness of the contour C(t): from the expression for the shearing force on the fixed contour (Eq. (12.9) from [2])

$$F=-\frac{EH^3}{12(1-\nu^2)}\left(\frac{\partial^3 u}{\partial n^3}+k\frac{\partial^2 u}{\partial n^2}\right)$$

it follows that, as a consequence of (1.3), at corner points there would be a concentrated reaction; this is impossible for a restrained, slightly flexed plate.

As an example, we consider the case of propagation of a circular crack, whose surface is free of load everywhere except at the center, while at the center the distance 2h between opposite edges increases at the constant rate 2U. Making the substitution

$$u(x, y, t)=Ut f(\xi), \qquad \xi=\frac{r^2}{a_1 t}, \qquad r^2=x^2+y^2$$

in (1.1)-(1.3) and solving the ordinary differential equation obtained, we find equations for the radius of the crack R and the displacement u:

$$R^2=\lambda\left(\frac{U}{a_1A_1}\right)a_1 t, \qquad \frac{U}{a_1A_1}=\frac{\lambda}{4}\,\frac{\operatorname{si}\lambda/4+\pi/2}{\sin\lambda/4}$$

$$f(\xi)=-\left(\operatorname{si}\frac{\lambda}{4}+\frac{\pi}{2}\right)^{-1}\left(\operatorname{si}\frac{\xi}{4}-\frac{\xi}{4}\operatorname{ci}\frac{\xi}{4}+\sin\frac{\xi}{4}+\frac{\xi\lambda}{4}\operatorname{ci}\frac{\lambda}{4}-\xi\sin\frac{\lambda}{4}-\operatorname{si}\frac{\lambda}{4}\right)$$

In these equations si, ci denote the integral sine and cosine.

For the case of infinitely slow motion by passing to the limit as $\lambda\to 0$ we obtain the radius of the equilibrium crack

$$R^2=2h/A_1$$

2. Let us formulate the problem of the development of a crack along a bar using Timoshenko's beam approximation [3] to describe the motion of the bar, i.e., taking into account the shear potential energy and the kinetic energy of rotational motion of cross sections of the bar. Let the axis of abscissas be directed along the bar; b and H are the transverse dimensions of one half of the bar, and $I = bH^3/12$ is the static moment of inertia of the cross section of that half. The crack is located on the interval $0 \le x \le l(t)$. At $x = 0$ we assign a transverse load F(t) and bending moment M(t) as functions of time, while at $x = l(t)$ the beam is rigidly clamped. We represent the slope of the neutral axis $\partial u/\partial x$ as the sum of a rotation ω and shear γ and write the kinetic and potential energies:

$$K=\int_0^l\left[\frac{\rho bH}{2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{I\rho}{2}\left(\frac{\partial\omega}{\partial t}\right)^2\right]dx$$

$$\Pi=\int_0^l\left[\frac{EI}{2}\left(\frac{\partial\omega}{\partial x}\right)^2+\frac{GbH}{2}\gamma^2\right]dx+Tbl(t)-F(t)\,u(0,t)-M(t)\,\omega(0,t) \tag{2.2}$$

$$G=\frac{E}{2(1+\nu)}$$

We find the equations of motion and the necessary boundary conditions from the condition of stationarity of the action integral for conditions of rigid restraint at the end of the crack:

$$u(l)=0,\ \omega(l)=0 \tag{2.1}$$

We present the result:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial\omega}{\partial x}=\frac{1}{c_2^2}\frac{\partial^2 u}{\partial t^2}, \qquad c_2^2=\frac{G}{\rho}, \quad \frac{\partial^2\omega}{\partial x^2}+\frac{c_2^2}{a^2}\left(\frac{\partial u}{\partial x}-\omega\right)=\frac{1}{c_0^2}\frac{\partial^2\omega}{\partial t^2}, \qquad a^2=\frac{c_0^2H^2}{12}, \qquad c_0^2=\frac{E}{\rho} \tag{2.2}$$

$$GbH\left(\frac{\partial u}{\partial x}-\omega\right)=F(t), \qquad EI\frac{\partial\omega}{\partial x}=M(t) \quad \text{at} \quad x=0 \tag{2.3}$$

$$\left(\frac{\partial u}{\partial x}\right)^2\frac{c_2^2}{a^2}\left[1-\frac{1}{c_2^2}\left(\frac{dl}{dt}\right)^2\right]+\left(\frac{\partial\omega}{\partial x}\right)^2\left[1-\frac{1}{c_0^2}\left(\frac{dl}{dt}\right)^2\right]=\frac{2Tb}{EI}=A^2 \qquad \text{at} \quad x=l(t) \tag{2.4}$$

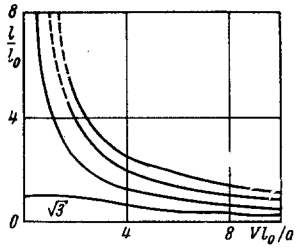

Fig. 1

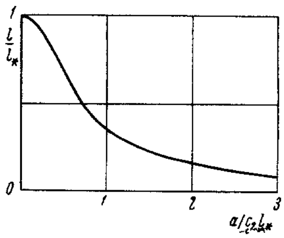

Fig. 2

Equations (2.2) coincide with the equations for [l], [m] from [3]. The first and second of these equations express Newton's second law for the translational and rotational motions of an element of the bar. By means of differentiations we can eliminate ω from (2.2) and obtain a fourth-order equation for u(x, t); however, this will not work with the boundary conditions. Since the motion of the crack is accompanied by the simultaneous motion of two beams, in order to exclude the possibility of overlapping of the edges of the crack it is necessary to require that the displacement does not change sign at $0 \leq x \leq l$. From (2.4) there follows the impossibility of propagation of the crack at a velocity exceeding the longitudinal wave velocity. When the Bernoulli-Euler approximation is employed, the velocity of the crack may be arbitrarily large [1].

Consider the steady cleaving action of a wedge of thickness 2h moving at velocity V. In this case

$$u(x, t) = u(x - Vt),$$
$$\omega(x, t) = \omega(x - Vt), \quad dl / dt = V \quad \text{at} \quad x = Vt$$
$$u = h, \ \partial\omega / \partial x = 0, \quad \text{at} \quad x = Vt + l$$

conditions (2.1) and (2.4) are satisfied. After substituting $\chi = x - Vt$ we easily find $u(\chi)$, $\omega(\chi)$;

$$u(\chi) = \frac{C \sin \alpha\chi}{\alpha^2 (1 - \beta_2^2)} + \frac{C\chi}{\alpha} \cos \alpha l + h, \qquad \omega(\chi) = \frac{C}{\alpha} (\cos \alpha l - \cos \alpha\chi)$$
$$C = \frac{h\alpha^2 (1 - \beta_2^2)}{\sin \alpha l - (1 - \beta_2^2) \alpha l \cos \alpha l}, \qquad \alpha = \frac{V}{a} (1 - \beta_2^2)^{-1/2} (1 - \beta_0^2)^{-1/2}$$
$$\beta_2 = V / c_2, \ \beta_0 = V / c_0$$

Using (2.4), we obtain a relation between the cleavage rate and the length of the crack in front of the wedge

$$\frac{[\sin \alpha l - (1 - \beta_2^2) \alpha l \cos \alpha l] (1 - \beta_0^2)^{1/2}}{[1 - (1 - \beta_2^2) \cos^2 \alpha l]^{1/2}} = \frac{h c_2^2 \beta_2^2}{a^2 A} \tag{2.5}$$

If the shear stiffness is infinte ($\beta_2 = 0$) and the propagation velocity is small ($\beta_0 \ll 1$), so that rotational inertia does not play an important part, then $\alpha \approx V/a$ and (2.5) gives the solution of the wedge problem in the Bernoulli-Euler approximation

$$1 - \frac{Vl_*}{a} \frac{l}{l_*} \operatorname{ctg} \left(\frac{Vl_*}{a} \frac{l}{l_*} \right) = \frac{1}{3} \left(\frac{Vl_*}{a} \right)^2, \qquad l_*^2 = \frac{3h}{A} \tag{2.6}$$

where l_* is the equilibrium length of the crack in the Bernoulli-Euler approximation [see [1], Eq. (2.5)].

In this case l/l_* is determined by only one dimensionless parameter Vl_*/a. Relation (2.6) is shown in Fig. 1. The dashed lines represent solutions for which for at least one point χ lying between 0 and l, $u(\chi) < 0$ and which, consequently, cannot be used to describe the motion of the crack. Unfortunately, in [1] the multivaluedness of the length in the case of steady wedge action was overlooked and only the graph passing through $l/l_* = 1$ was presented.

Let us now consider slow motions, assuming that the bar has a finite shear stiffness. As before, we take $\alpha \approx V/a$; however, $\beta_2 \neq 0$. Replacing the trigonometric functions in (2.5) by segments of a Taylor series and neglecting high-order infinitesimals as $V \to 0$, we obtain an equation for the length of the crack l_0 in front of the stationary wedge:

$$l_0^6 + 6\frac{a^2}{c_2^2} l_0^4 + 9\left(\frac{a^4}{c_2^4} - \frac{h^2}{A^2}\right) l_0^2 = \frac{9h^2a^2}{A^2c_2^2} \tag{2.7}$$

Dividing (2.7) by l_*^6, we obtain the relationship between l_0/l_* and $a/c_2 l_*$. This relationship, which makes it possible to determine when the shear potential energy can be disregarded, is shown in Fig. 2. As $a/c_2 l_* \to \infty$ the length of the crack tends to zero.

For large cleavage rates, we find that motions with velocities $V > c_2$ are impossible, since the left-hand side of (2.5) becomes purely imaginary at $\beta_2 > 1$. Investigation shows that for each velocity $V < c_2$ there is an infinite set of crack lengths in front of the wedge, so that the graph of $l(V)$ has an infinite set of branches. If $hc_2^2/a^2A \leq (1 - c_2^2/c_0^2)^{1/2}$, all the branches lie on the interval $0 \leq V \leq c_2$ and pass through the point $l = 0$, $V = c_2$. However, if $hc_2^2/a^2A > (1 - c_2^2/C_0^2)^{1/2}$, then all the branches are cut off at $V < c_2$, and the higher the branch, the more closely it approaches the point $l = 0$, $V = c_2$.

In the case considered the choice of a particular crack length is determined by the initial conditions.

It should not be concluded from this example that c_2 in the Timoshenko approximation is the limiting velocity for motion of the crack under any conditions. To convince oneself of this it is sufficient to consider the steady-state propagation of a crack activated by a concentrated moment exceeding $\sqrt{2TbEI}$.

LITERATURE CITED

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